τ add. char. of $\mathbb{A}/k$. number field k $G = GL_2$

π irred. admissible $\mathcal{H}(\mathbb{A})$-mod.

$\pi \simeq \bigotimes_v' \pi_v$ Assume π_v has a Whittaker model. $\mathcal{W}_{\tau_v}(\pi_v)$

nontriv. add. char.

π_v adm. $\mathcal{H}(k_v)$-mod.

$\mathcal{H}(k_v) = \left(C_c^\infty(G(k_v)) \quad v \text{ non-arch} \right)$

$\rightsquigarrow$ local zeta function

$$Z_v(s, W_v, \chi_v) = \int_{k_v^\times} W_v\left(\begin{pmatrix} x & \\ & 1 \end{pmatrix}\right) \chi_v(x) |x|_v^{s-\frac{1}{2}} d^\times x$$

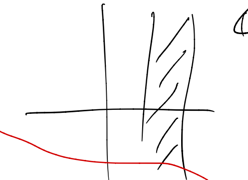
$W_v \in \mathcal{W}_{\tau_v}(\pi_v)$, χ_v char of $k_v^\times$. χ char of $\mathbb{A}^\times/k^\times$

g.c.d. $\rightsquigarrow$ $L(s, \pi_v \times \chi_v)$ global $\rightsquigarrow$ $L(s, \pi \times \chi) = \prod_v L(s, \pi_v \times \chi_v)$

local F.E. $\rightsquigarrow$ $\varepsilon(s, \pi_v \times \chi_v, \tau_v)$ $\quad \varepsilon(s, \pi \times \chi) = \prod_v \varepsilon(s, \pi_v \times \chi_v, \tau_v)$

Thm 4 (see Alex's talk)

π irred. cuspidal automorphic w/ central char. ω. $\left(\pi \subseteq L_0^2(G(k)\backslash G(\mathbb{A}), \omega)\right)$

$\Rightarrow$ $L(s, \pi \times \chi)$ has analytic continuation to all s 

$\Leftarrow$??

bounded on every vertical strip

has functional equation $L(1-s, \pi \times \omega^{-1}\chi^{-1})$

$$L(s, \pi \times \chi) = \varepsilon(s, \pi \times \chi) L(1-s, \check{\pi} \times \check{\chi})$$

$\forall$ char χ of $\mathbb{A}^\times / k^\times$. $\check{\pi} \simeq \pi \otimes \omega^{-1} \circ \det$ for GL_2.

Nice

Converse thm:

$\pi = \otimes' \pi_v$ irred. adm. $\mathcal{H}(\mathbb{A})$-mod. w/ central char ω of $\mathbb{A}^\times / k^\times$.

$\pi_v^{K_v} \neq 0$ $\forall$ a.e. v, π_v has Whittaker model.

$L(s, \pi \times \chi)$ "nice" $\forall \chi$ of $\mathbb{A}^\times / k^\times$

$\Rightarrow$ $\pi \subseteq L_0^2(G(k)\backslash G(\mathbb{A}), \omega)$

Idea of Proof: $\pi \simeq \bigotimes_v' \pi_v$

$$W_\tau(\pi) \simeq \bigotimes_v' W_{\tau_v}(\pi_v)$$

Construct a non-zero intertwining map

$$W_\tau(\pi) \longrightarrow L^2_0(G(k)\backslash G(\mathbb{A}), \omega)$$

$$W \longmapsto \boxed{g \mapsto \sum_{\xi \in k^\times} W\left(\begin{pmatrix} \xi & \\ & 1 \end{pmatrix} g\right)} = \varphi_W(g)$$

Need to show:

φ_W is ① abs. conv. (b/c Whittaker functions are nice)

② rapidly decreasing ($\Rightarrow L^2$)

③ cuspidal (b/c no constant term)

④ left $\begin{pmatrix} * & * \\ & * \end{pmatrix}$ - inv. (easy)

⑤ left $\begin{pmatrix} & 1 \\ -1 & \end{pmatrix}$ - inv. ← F.E.

④, ⑤ $\Rightarrow$ left $G(k)$-inv.

Focus on ⑤.

$$Z(s, \pi(g)W, \chi) = \int_{\mathbb{A}^\times} W\left(\begin{pmatrix} x & \\ & 1 \end{pmatrix} g\right) \chi(x) |x|^{s-\frac{1}{2}} d^\times x$$

$$= \int_{k^\times \backslash \mathbb{A}^\times} \sum_{\xi \in k^\times} W\left(\begin{pmatrix} \xi x & \\ & 1 \end{pmatrix} g\right) \chi(x) |x|^{s-\frac{1}{2}} d^\times x$$

$$= \int_{k^\times \backslash \mathbb{A}^\times} \varphi_W\left(\begin{pmatrix} x & \\ & 1 \end{pmatrix} g\right) \chi(x) |x|^{s-\frac{1}{2}} d^\times x$$

Mellin transform of φ_W.

$$Z(s, \pi(g)W, \chi) = \prod_v Z(s, \pi_v(g_v) W_v, \chi_v)$$

$$= L(s, \pi \times \chi) \underbrace{\prod_{v \in S}}_{\text{fin. prod.}} \frac{Z_v(s, \pi_v(g_v) W_v, \chi_v)}{L(s, \pi_v \times \chi_v)}$$

Similarly $w = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}$

$$Z(1-s, \pi(wg)W, \omega^{-1}\chi^{-1}) = L(1-s, \underbrace{\pi \times \omega^{-1}\chi^{-1}}_{\check{\pi} \times \check{\chi}}) \prod_{v \in S} \frac{Z_v(1-s, \pi_v(wg_v) W_v, \omega_v^{-1}\chi_v^{-1})}{L(s, \check{\pi}_v \times \check{\chi}_v)}$$

By F.E. on L-fun $\Rightarrow$ $Z(s, \pi(g)W, \chi) = Z(1-s, \pi(wg)W, \omega^{-1}\chi^{-1})$.

Ignore convergence

set $s = \frac{1}{2}$

$Z(\frac{1}{2}, \pi(g)W, \chi)$ Mellin transf. of $\varphi_W\left(\begin{pmatrix} \cdot & \\ & 1\end{pmatrix} g\right)$ w.r.t. χ

$Z(\frac{1}{2}, \pi(wg)W, \omega^{-1}\chi^{-1})$ $\varphi_W\left(w\begin{pmatrix} \cdot & \\ & 1\end{pmatrix} g\right)$ w.r.t. χ

$$\begin{aligned}
&\| \\
&\int_{k^\times\backslash\mathbb{A}^\times} \varphi_W\left(\begin{pmatrix} x & \\ & 1\end{pmatrix} wg\right) \omega^{-1}(x)\chi^{-1}(x)\, d^\times x \\
&= \int \varphi_W\left(w\begin{pmatrix} 1 & \\ & x\end{pmatrix} g\right) \omega^{-1}(x)\chi^{-1}(x)\, d^\times x \\
&= \int \varphi_W\left(w\begin{pmatrix} x^{-1} & \\ & 1\end{pmatrix} g\right) \cancel{\omega(x)}\, \cancel{\omega^{-1}(x)}\chi^{-1}(x)\, d^\times x \\
&= \int \varphi_W\left(w\begin{pmatrix} x & \\ & 1\end{pmatrix} g\right) \chi(x)\, d^\times x
\end{aligned}$$

$\Downarrow$ Inverse Mellin transform

$\varphi_W\left(\begin{pmatrix} x & \\ & 1\end{pmatrix} g\right) = \varphi_W\left(w\begin{pmatrix} x & \\ & 1\end{pmatrix} g\right)$

$\forall x, g,$

$\Rightarrow \varphi_W$ is left-inv. by w.

$\square$